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Relaxation methods and coupling procedures

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SUMMARY

This paper studies a global relaxation method to ensure the conservative coupling at a fixed interface of two Euler systems with different pressure laws. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

What we consider as our coupling problem is, given two different systems,

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}_L(\mathbf{u}) = \mathbf{0}, \quad x<0, \qquad \partial_t \mathbf{u} + \partial_x \mathbf{f}_R(\mathbf{u}) = \mathbf{0}, \quad x>0, \qquad t>0 \tag{1}$$

where $\mathbf{f}_\alpha, \alpha = L, R$, are two smooth fluxes, find a function $\mathbf{u}$ satisfying (1) with some initial condition $\mathbf{u}(x,0) = \mathbf{u}_0(x), x \in \mathbb{R}$, and a coupling condition (CC) at the interface $x = 0$. A new CC was introduced in [1] and followed by applications to problems arising in the frame of multiphase flow simulations [2], with the further aim of coupling codes. The condition says that two boundary value problems should be well posed, and results in imposing whenever possible the continuity of the solution $\mathbf{u}$ at the interface, without imposing the conservativity of the coupled model, what we call *state coupling* as opposed to *flux coupling*. *Flux coupling* appears as a conservative approach of a system with discontinuous flux, since we can then write (1) for $x \in \mathbb{R}$, with a unique flux $\mathbf{f}(\mathbf{u}, x) = (1 - H(x))\mathbf{f}_L(\mathbf{u}) + H(x)\mathbf{f}_R(\mathbf{u})$, where H is the Heaviside function. We will see that, in the case we consider below, we can also achieve this conservative coupling through the *state coupling* of larger relaxation systems. Indeed, we have introduced in [3] a *global relaxation approximation*

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involving relaxation systems and a state coupling procedure as a tool for *flux coupling* of two Euler systems at a fixed interface.

First, recall that the *state coupling* approach was generalized in [2, 4] by introducing the possibility of transmitting other variables, say **v**, than the conservative ones **u**. In the case of Euler system in Lagrangian coordinates, we may choose for **v** the primitive variables and transmitting the velocity and the pressure eventually results in flux coupling. However, the flux variables cannot always be taken as transmitted variables. Indeed in the case of the Euler system (2), the problem is more delicate since the eigenvalues may vanish and change the sign and this might create a resonance phenomenon at the interface [5]. Here comes the idea of using a relaxation method which, by introducing a larger but simpler system on which a state coupling procedure is applied, ensures a conservative coupling of two Euler systems which avoids resonance. Then state and flux coupling can be combined in an optimized hybrid coupling. We are presently extending the approach to more complex systems (bifluid models).

In Section 2, we introduce the relaxation system. In Section 3, we study the coupled Riemann problem (CRP) of two linearly degenerate systems involved in the global relaxation approximation which we present in Section 4. Numerical tests illustrate the good properties of this global relaxation solver.

2. THE RELAXATION SYSTEM

The systems involved in (1) will be two Euler systems differing by their pressure law

$$\begin{aligned} &\partial_t \varrho + \partial_x(\varrho u) = 0,\\ &\partial_t(\varrho u) + \partial_x(\varrho u^2 + p) = 0,\\ &\partial_t(\varrho e) + \partial_x((\varrho e + p)u) = 0, \end{aligned} \qquad p = \begin{cases} p_L(\varrho, \varepsilon) = \tilde{p}_L(\tau, s) & \text{in } x<0\\ p_R(\varrho, \varepsilon) = \tilde{p}_R(\tau, s) & \text{in } x>0 \end{cases} \tag{2}$$

where p_α is a given function expressed either in terms of density and internal energy (ϱ, ε) or equivalently (with a tilde) in terms of specific volume and entropy (τ, s) (with $\tau = 1/\varrho$ and the physical specific entropy is $-s$). We assume that these functions satisfy the classical assumptions derived from laws of thermodynamics, in particular we can write $\varepsilon = \tilde{\varepsilon}(\tau, s)$ with $\tilde{\varepsilon}_\tau = -p < 0$, $\tilde{\varepsilon}_s = -T < 0$, and $\tilde{\varepsilon}(\tau, s)$ strictly convex. The internal energy satisfies the relation $e = \varepsilon + u^2/2$, the set of states is $\Omega^E = \{(\varrho, q = \varrho u, E = \varrho e);\ \varrho > 0, u \in \mathbb{R}, e - u^2/2 > 0\}$. Outside coupling (when $p_L = p_R$), discontinuous solutions of (2) are required to satisfy an inequality $\partial_t \varrho s + \partial_x(\varrho s u) \leqslant 0$, which becomes an equality for smooth solutions

$$\partial_t \varrho s + \partial_x(\varrho s u) = 0 \tag{3}$$

Following [6], we propose to approximate the solutions of (2) by those of the relaxation system

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0,\\ &\partial_t(\rho u) + \partial_x(\rho u^2 + \Pi) = 0,\\ &\partial_t \varrho s + \partial_x(\varrho s u) = 0,\\ &\partial_t(\rho \mathscr{T}) + \partial_x(\rho \mathscr{T} u) = \lambda \rho(\tau - \mathscr{T}), \end{aligned} \qquad \Pi = \begin{cases} \tilde{p}_L(\mathscr{T}, s) + a^2(\mathscr{T} - \tau) & \text{in } x<0\\ \tilde{p}_R(\mathscr{T}, s) + a^2(\mathscr{T} - \tau) & \text{in } x>0 \end{cases} \tag{4}$$

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with a singular perturbation where $\lambda>0$ stands for the relaxation parameter. The set of states for (4) is $\Omega=\{\mathbf{U}=(\varrho, q=\varrho u, S=\varrho s, \mathscr{I}=\varrho\mathscr{T}); \varrho>0, u\in\mathbb{R}, s>0, \mathscr{T}>0\}$. The closure relation for Π is the exact pressure law $\tilde{p}$ modified by a correction term, where a is a positive constant which is required to satisfy Whitham's or subcharacteristic condition

$$a^2>-\partial_\tau\tilde{p}_\alpha(\mathscr{T}, s) \tag{5}$$

i.e. to upper bound the exact Lagrangian sound speed for all the states $\mathbf{U}$ under consideration. Formally, as the relaxation parameter $\lambda\to\infty$, $\mathscr{T}-\tau\to 0$, so that $\Pi\to p$ and we indeed obtain Euler system at equilibrium where the energy equation has been replaced by entropy (3). When $p_L=p_R$, outside coupling, a proof of the convergence for smooth solutions of (4) to smooth solutions of (2) can be found in [7]. It is then shown in [8] how to recover a consistent approximation of weak solutions of (2). The relaxation procedure is justified by an entropy dissipation principle (see [6]) involving the energy $\mathscr{E}=\varrho\Sigma$ and energy flux $\mathscr{G}_\mathscr{E}$ where

$$\Sigma(\mathbf{U})=\tilde{\varepsilon}(\mathscr{T}, s)+u^2/2+(\tilde{\Pi}^2(\tau, s, \mathscr{T})-\tilde{p}^2(\mathscr{T}, s))/2a^2, \quad \mathscr{G}_\mathscr{E}(\mathbf{U})=(\varrho\Sigma+\Pi)u \tag{6}$$

System (4) without source term ($\lambda=0$) has four linearly degenerate fields $u-a\tau, u, u, u+a\tau$, and explicit solutions for Riemann problems, which we note $\mathbf{W}_\alpha(x/t; \mathbf{U}_\ell, \mathbf{U}_r)$ when the law p_α is used on the whole line (without coupling).

3. COUPLED RIEMANN PROBLEM

The numerical method we describe in Section 4 involves the coupling, at a fixed interface $x=0$, of two such systems (4) with $\lambda=0$, differing by their pressure law. We now study the CRP, which corresponds to initial data $\mathbf{U}_\ell$ for $x<0$ and $\mathbf{U}_r$ for $x>0$ and with a CC which we now describe. Since we want to transmit, not the conservative variables but the primitive variables $\mathbf{V}=(\tau, u, \Pi, \Sigma)^{\mathrm{T}}$, we introduce the law $\mathscr{T}_\alpha(\tau, \Pi, s)$ obtained by inverting the relation $\tilde{p}_\alpha(\mathscr{T}, s)+a^2\mathscr{T}=\Pi+a^2\tau$, for a satisfying (5) and two associated changes of variables

$$\mathbf{U}=\varphi_\alpha(\mathbf{V}) \quad \text{where } \varphi_\alpha(\tau, u, \Pi, \Sigma)=(\varrho, \varrho u, \varrho s, \varrho\mathscr{T}_\alpha(\tau, \Pi, s)) \tag{7}$$

Note that (6) yields $\partial_s\Sigma(\tau, u, \Pi, s)=\partial_s\tilde{\varepsilon}(\mathscr{T}, s)<0$ by assumption so that φ_α is indeed an admissible change of variables. The CC with *transmission* of $\mathbf{V}$ writes

$$\begin{aligned}&\mathbf{U}(0-, t)\in\mathscr{O}_L(\varphi_L(\mathbf{V}(0+, t))) \quad \forall t>0\\ &\mathbf{U}(0+, t)\in\mathscr{O}_R(\varphi_R(\mathbf{V}(0-, t))) \quad \forall t>0\end{aligned} \tag{8}$$

where the sets $\mathscr{O}_\alpha$ are sets of traces of possible Riemann problems with one given data $\mathscr{O}_L(\mathbf{U}_b)=\{\mathbf{W}_L(0-; \mathbf{U}, \mathbf{U}_b), \mathbf{U}\in\Omega\}$, $\mathscr{O}_R(\mathbf{U}_b)=\{\mathbf{W}_R(0+; \mathbf{U}_b, \mathbf{U}), \mathbf{U}\in\Omega\}$ (we refer to [4] for details). Relations (8) say that the two boundary value problems in each half space $t>0$, $x>0$, or $x<0$, are well posed (cf. [9]). However, the boundary data are characterized in terms of primitive (as in [4]), not conservative variables (as they were in [1]). In [4] for the case of two Lagrangian systems we solved directly (1) with $\mathbf{u}=(\tau, v, e)^{\mathrm{T}}, \mathbf{f}_\alpha(\mathbf{u})=(-v, p, pv)^{\mathrm{T}}$, $p=p_\alpha$, where the CC in variables $\mathbf{v}=(\tau, v, p)^{\mathrm{T}}$, eventually resulted in the continuity of the flux. Here also, for coupling the two systems ($4, \lambda=0$), $\mathbf{V}$ is a variable for the CC such that the flux is continuous. Given data *at equilibrium*, i.e. such that $\mathscr{T}_\alpha=\tau$, $\alpha=L, R$, we prove the following.

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Theorem 1
Assuming (5), for any given equilibrium states $\mathbf{U}_L, \mathbf{U}_R$, the CRP (4,$\lambda=0$), with CC (7), (8) admits a unique solution $\mathbf{W}_c^{\mathbf{V}}(x/t; \mathbf{U}_L, \mathbf{U}_R)$ which coincides with the solution of the classical Riemann problem in each half space

$$\mathbf{W}_c^{\mathbf{V}}(x/t; \mathbf{U}_L, \mathbf{U}_R) = \begin{cases} \mathbf{W}_L(x/t; \mathbf{U}_L, \varphi_L(\mathbf{V}_R)), & x<0 \\ \mathbf{W}_R(x/t; \varphi_R(\mathbf{V}_L), \mathbf{U}_R), & x>0 \end{cases} \tag{9}$$

Moreover, ϱu, $\varrho u^2+\Pi$ and $(\varrho\Sigma+\Pi)u$, are continuous at the interface.

We prove that the solution we obtain for the above CRP expressed in terms of $\mathbf{V}$ coincides with the solution of the Riemann problem of the conservative system in variables $(\varrho, \varrho u, \varrho\Pi, \varrho\Sigma)$, where we have replaced the third equation in (4) by $\partial_t(\varrho\Pi)+\partial_x((\varrho\Pi+a^2)u)=0$, which is easy to verify and the last one by the energy conservation equation. Since all the characteristic fields are linearly degenerate, the two systems are equivalent. If we express the solution in terms of $\mathbf{U}$, we have to use the two changes of variables $\mathscr{T}_\alpha$ (as in (7)) and s.

4. GLOBAL RELAXATION APPROXIMATION

Let us now define our global relaxation approximation that involves a relaxation solver and a coupling solver. Recall that this numerical procedure is devised to approximate the conservative coupling of two Euler systems (2).

4.1. The relaxation solver

We use a fractional step method to advance the solution in time from t_n to $t_{n+1}=t_n+\Delta t$. Three steps are needed in the relaxation solver: reconstruction, evolution, projection, the last two steps correspond to solve the relaxation system (4) by an operator splitting method [8]. We first describe the main lines of the relaxation procedure for $n=0$.

1. Let $\mathbf{u}_0(x)=(\varrho_0, \varrho_0 u_0, \varrho_0 e_0)^{\mathrm{T}}(x)$ be an initial datum for system (2). We define the extended initial datum at equilibrium $\mathbf{U}_0(x)=(\varrho_0, \varrho_0 u_0, \varrho_0 s_0, \varrho_0 \mathscr{T}_0)^{\mathrm{T}}(x)$ for system (4) with $\mathscr{T}_0 \equiv 1/\varrho_0$, $s_0=s(\varrho_0, \varepsilon_0)$.
2. We solve (approximately) the coupled Cauchy problem for (4) with $\lambda=0$, and $\mathbf{V}$—CC at $x=0$, we obtain $\mathbf{U}_1^-(x)=\mathbf{U}(x, \Delta t)$.
3. We project $\mathbf{U}_1^-=(\varrho_1, \varrho_1 u_1, \varrho_1 s_1, \varrho_1 \mathscr{T}_1)^{\mathrm{T}}$ on the equilibrium set of system (4) to obtain $\mathbf{U}_1=(\varrho_1, \varrho_1 u_1, \varrho_1 s_1, 1)^{\mathrm{T}}$ (instantaneous relaxation).

4.2. Definition of the coupling solver

In step 2, we need a numerical coupling procedure which we recall briefly (see [4]). Let Δx denote the space step, we set $\mu=\Delta t/\Delta x$, and consider the cells $C_{j+1/2}=(x_j, x_{j+1})$, with center $x_{j+1/2}=(j+1/2)\Delta x$, $j\in\mathbb{Z}$. The initial condition is discretized as usually by

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$\mathbf{U}^0_{j+1/2}=(1/\Delta x)\int_{C_{j+1/2}}\mathbf{U}_0(x)\,\mathrm{d}x$. We are given two numerical fluxes $\mathbf{G}_L,\mathbf{G}_R$ ($\mathbf{G}_\alpha$ consistent with $\mathbf{F}_\alpha$) corresponding to 3-point schemes, and we define the coupling scheme by

$$\begin{aligned}\mathbf{U}^{n+1}_{j-1/2}&=\mathbf{U}^n_{j-1/2}-\mu(\mathbf{G}^n_{L,j}-\mathbf{G}^n_{L,j-1}),\quad j\leqslant 0,\ \ n\geqslant 0\\ \mathbf{U}^{n+1}_{j+1/2}&=\mathbf{U}^n_{j+1/2}-\mu(\mathbf{G}^n_{R,j+1}-\mathbf{G}^n_{R,j}),\quad j\geqslant 0,\ \ n\geqslant 0\end{aligned}\tag{10}$$

So we have usual fluxes for $j\neq 0$ and two fluxes $\mathbf{G}^n_{\alpha,0}$ at the interface at $x=0$. If we take for $\mathbf{G}_{\alpha,j}$, $j\neq 0$ Godunov's flux (noted with a superscript God) and, use the solution of the CRP, according to the coupling procedure

$$\mathbf{G}^{\mathrm{God},n}_{L,0}=\mathbf{F}_L(\mathbf{W}^{\mathbf{V}}_c(0-;\mathbf{U}^n_{-1/2},\mathbf{U}^n_{+1/2})),\quad \mathbf{G}^{\mathrm{God},n}_{R,0}=\mathbf{F}_R(\mathbf{W}^{\mathbf{V}}_c(0+;\mathbf{U}^n_{-1/2},\mathbf{U}^n_{+1/2}))\tag{11}$$

we obtain what we call the Godunov scheme with $\mathbf{V}$ state coupling

$$\begin{aligned}\mathbf{U}^{n+1}_{j-1/2}&=\mathbf{U}^n_{j-1/2}-\mu(\mathbf{G}^{\mathrm{God},n}_{L,j}-\mathbf{G}^{\mathrm{God},n}_{L,j-1}),\quad j\leqslant 0,\ \ n\geqslant 0\\ \mathbf{U}^{n+1}_{j+1/2}&=\mathbf{U}^n_{j+1/2}-\mu(\mathbf{G}^{\mathrm{God},n}_{R,j+1}-\mathbf{G}^{\mathrm{God},n}_{R,j}),\quad j\geqslant 0,\ \ n\geqslant 0\end{aligned}\tag{12}$$

4.3. The global relaxation solver

We now define the resulting global relaxation scheme when we use Godunov's flux in each half space. Given an initial data for (2) $\mathbf{u}_0$, define $\mathbf{u}^0_{j+1/2}=(1/\Delta x)\int_{C_{j+1/2}}\mathbf{u}_0(x)\,\mathrm{d}x$, then from $\mathbf{u}^n_{j+1/2}=(\varrho,\varrho u,\varrho e)^n_{j+1/2}$, define $\mathbf{U}^n_{j+1/2}=((\varrho,\varrho u,\varrho s)^n_{j+1/2},1)$, $j\in\mathbb{Z}$.

- Solve the Riemann problems, compute $\mathbf{W}_\alpha(0;\mathbf{U}^n_{j-1/2},\mathbf{U}^n_{j+1/2})$, $\alpha=L$, $j<0$, $\alpha=R$, $j>0$, to define the numerical fluxes $\mathbf{G}^{\mathrm{God},n}_{\alpha,j}$, $j\neq 0$. Solve the CRP with data $\mathbf{U}^n_{-1/2},\mathbf{U}^n_{+1/2}$ and prescribed state CC, compute $\mathbf{W}^{\mathbf{V}}_c(x/t;\mathbf{U}^n_{-1/2},\mathbf{U}^n_{+1/2})$ and (11).
- Update $\mathbf{U}^{n+1}_{j+1/2}$ by (12), (11); keep the two first components that provide a scheme consistent with the equations of conservation of mass and momentum, the third one gives an equation for the entropy, with numerical flux at the interface $x=x_j$ noted $\mathscr{G}^n_{\alpha,\varrho s,j}$, $\alpha=L$ or R.
- A careful analysis as in [8] enables us to reverse the role of entropy conservation and energy inequality and turn back to a scheme with an equation for the energy that satisfies an entropy inequality. We skip the technical details for which we refer to [10].

We thus assert that with $\mathbf{V}$-state coupling, the resulting global Godunov relaxation solver is conservative in $(\varrho,\varrho u,\varrho e)$:

Theorem 2
The resulting scheme can be expressed as

$$\mathbf{u}^{n+1}_{j+1/2}=\mathbf{u}^n_{j+1/2}-\mu(\mathbf{g}^n_{j+1}-\mathbf{g}^n_j),\quad j\in\mathbb{Z},\ \ n\geqslant 0\tag{13}$$

with the three components of the flux $\mathbf{g}^n_j$ given by $(\varrho u,\varrho u^2+\Pi,(\varrho\Sigma+\Pi)u)(\mathbf{W}^n_j)$ (i.e. evaluated on state $\mathbf{W}^n_j$)

$$\mathbf{W}^n_j=\begin{cases}\mathbf{W}_\alpha(0;\mathbf{U}^n_{j-1/2},\mathbf{U}^n_{j+1/2}), & \alpha=L\ \text{ if } j<0,\quad \alpha=R\ \text{ if } j>0\\ \mathbf{W}^{\mathbf{V}}_c(0;\mathbf{U}^n_{-1/2},\mathbf{U}^n_{+1/2}), & j=0\end{cases}\tag{14}$$

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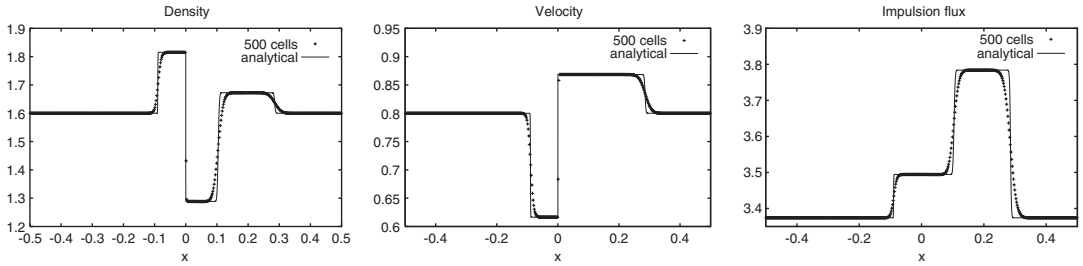

Figure 1. Test 1: density, velocity and impulsion flux.

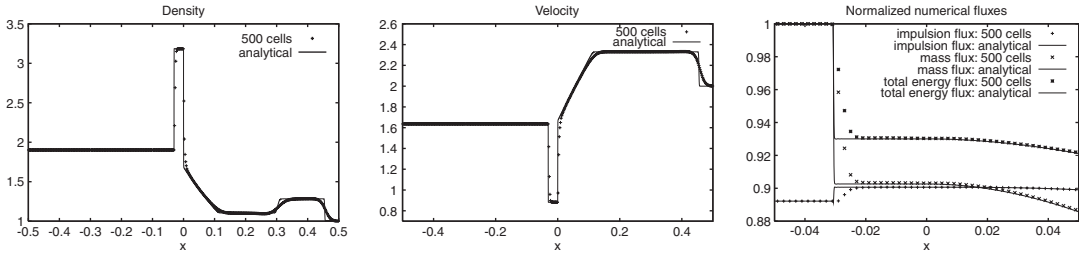

Figure 2. Test 2: density, velocity and numerical flux components.

Moreover, the global Godunov relaxation solver satisfies a discrete entropy inequality

$$\begin{aligned}(\varrho s)^{n+1}_{j+1/2} &\leqslant (\varrho s)^{n}_{j+1/2} - \mu(\mathscr{G}^{n}_{L,\varrho s,j+1} - \mathscr{G}^{n}_{L,\varrho s,j}), \quad j<0, \ \ n\geqslant 0\\ (\varrho s)^{n+1}_{j+1/2} &\leqslant (\varrho s)^{n}_{j+1/2} - \mu(\mathscr{G}^{n}_{R,\varrho s,j+1} - \mathscr{G}^{n}_{R,\varrho s,j}), \quad j\geqslant 0, \ \ n\geqslant 0\end{aligned} \tag{15}$$

Note that in (15), $(\varrho s)^{n+1}_{j+1/2}$ is defined from $s(\varrho^{n+1}_{j+1/2}, \varepsilon^{n+1}_{j+1/2})$ (and not by the third component of $\mathbf{U}^{n+1}_{j+1/2}$). We have moreover the formal Lax–Wendroff-type convergence result, where $\mathbf{u}_\Delta$ denotes the piecewise constant function associated in a classical way with scheme (13).

Theorem 3
Assume that $\mathbf{u}_\Delta$ is bounded in $\mathbb{L}^\infty$ and that the scheme converges in the sense that $\mathbf{u}_\Delta \to \mathbf{u}$ in $\mathbb{L}^1_{\mathrm{loc}}(\mathbb{R}+; \mathbb{L}^1_{\mathrm{loc}}(\mathbb{R}))$ and a.e. and $\mathbf{u}_\Delta(0,t) \to \mathbf{u}(0,t)$ in $\mathbb{L}^1_{\mathrm{loc}}(\mathbb{R})$ and a.e. Then the limit $\mathbf{u}$ is a weak solution of the conservative Euler system with pressure law $p = p(x) = (1-H(x))p_L + H(x)p_R$ and initial condition $\mathbf{u}_0$.

We illustrate our approach numerically with the solution of two CRP for the full Euler system (2) with two γ-laws. We take $\gamma_L = 1.4, \gamma_R = 1.6$ (CFL = 0.5, 500 points), we represent the density, velocity and numerical impulsion flux which is continuous at $x = 0$ (the interface is at the middle of the space domain $(-0.5, +0.5)$). For the first test, we start with a constant initial data $\varrho_L = \varrho_R = 1.6$, $u_L = u_R = 0.8$, $p_L = p_R = 2.35$. Because of the difference in pressure laws, it gives birth to three waves plus a standing wave and the exact solution is a $1L$-shock, a standing wave, a $2R$-contact discontinuity and a $3R$-rarefaction. These four waves are seen on the density (Figure 1, left),

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while the velocity (Figure 1, middle) is naturally constant on the $2R$-contact discontinuity, and the impulsion flux (Figure 1, right) is indeed continuous on the standing wave.

The second test is more delicate since it involves a sonic rarefaction wave (but no resonance occurs): $\varrho_L = 1.902, u_L = 1.6361, p_L = 2.4598; \varrho_R = 1, u_R = 2, p_R = 1$. The solution (shown at $t = 0.13$) is a $1L$-shock, a standing (coupling) wave, a sonic $1R$-rarefaction, a $2R$-contact discontinuity and a $3R$-shock, with a zoom on the three normalized numerical flux components (Figure 2) that are indeed continuous on the coupling wave (at $x = 0$). Note the presence of two waves of the same family (1-waves), due again to the difference of pressure laws hence of sound velocities at the interface. The solution does not depend on the mesh refinement (*computations by Thomas Galié*).

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